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DBK-003-1163002 Seat No. ________

M. Sc. (Sem. III) (CBCS) Examination

June – 2022

Mathematics : CMT - 3002

(Functional Analysis)

Faculty Code : 003

Subject Code : 1163002

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions :

(1) Attempt any **five** questions from the following.

(2) There are total **ten** questions.

(3) Each question carries **equal** marks.

1 Answer the following : **7×2=14**

(1) Let $T : X \to X$ be a linear transformation. Justify whether $R\ (T)$ is a vector space or not?

(2) Define with example: Continuous Linear transformation.

(3) Define with example: Banach Space.

(4) Justify whether a real valued function $f : [0,\ 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & when\ x \in [0,\ 1] \cap \mathbb{Q} \\ x, & when\ x \in [0,\ 1] \cap \mathbb{Q}^c \end{cases}$$

is essentially bounded or not?

(5) State Parseval's identity.

(6) Define with example: Weak* –Convergence.

(7) Define with example: Algebraic Dual Space.

2 Answer the following : **7×2=14**

(1) Justify whether dual space of l^{∞} is l^1 or not?

(2) Define with example : Sub-linear functional.

(3) Define with example: Direct Sum.

(4) Define with example: Hilbert space.

(5) Define with example: Orthogonal elements.

(6) Justify whether two Orthonormal elements of an Inner Product Space X are linearly independent or not?

(7) Define nowhere dense set. Give an example of uncountable set which is nowhere dense set.

3 Answer the following : **2×7=14**

(1) State and prove, Minkowski's Inequality.

(2) Let X and Y be two normed spaces.
Let $T : X \rightarrow Y$ be a linear transformation. Prove that, the following are equivalent :

a) T is continuous on X.

b) The null space $N(T)$ is closed in X and the linear transformation $\tilde{T} : X|_{N(T)} \rightarrow Y$ defined by

$$\tilde{T}\left(x+N(T)\right)=T(x),\ \forall x+N(T)\in X|_{N(T)} \text{ is continuous.}$$

4 Answer the following : **2×7=14**

(1) Prove that, every finite dimensional subspace of a normed space X is complete.

(2) Let $p \in [1, \infty)$. Prove that, l^p is a complete metric space.

5 Answer the following : **2×7=14**

(1) Prove that, on a finite dimensional vector space X, any norm $\|\cdot\|_a$ is equivalent to any other norm $\|\cdot\|_b$.

(2) Let X and Y be Normed linear space and let $B(X, Y)$ be the space of all bounded linear transformations from X into Y. If Y is a Banach space, prove that, $B(X, Y)$ is also a Banach space

6 Answer the following : **2×7=14**

(1) State and prove, Uniform Boundedness theorem.

(2) State Baire's Category theorem. Prove that, a Banach space does not have a countably infinite Hamel Basis.

7 Answer the following : **2×7=14**

(1) State and prove, closed graph theorem.

(2) State Hahn-Banach Theorem. Prove that, if X is any normed linear space over K then

$$\|x\| = \sup_{0 \neq f \in X'} \frac{|f(x)|}{\|f\|}, \ \forall x \in X.$$

8 Answer the following : **2×7=14**

(1) State and Prove, Projection Theorem.

(2) Let X be an Inner Product Space. Let $x_n \to x$ in X and $y_n \to y$ in X. Prove that, $\langle x_n, y_n \rangle \to \langle x, y \rangle$

9 Answer the following : **2×7=14**

(1) State and prove, Riesz-Representation Theorem.

(2) Prove that, every Hilbert space H is reflexive.

10 Answer the following : **2×7=14**

(1) State and prove, Parallelogram law as well as Pythagorean Relation.

(2) State and prove, Polarization identity.